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Hydrodynamic reductions of the dispersionless Harry Dym hierarchy

Jen-Hsu Chang

Department of Computer Science, Chung-Cheng Institute of Technology, National Defense University, Taoyuan, Taiwan

E-mail: jhchang@ccit.edu.tw

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Abstract

We investigate the reductions of the dispersionless Harry Dym hierarchy to systems of finitely many partial differential equations. These equations must satisfy the compatibility condition and they are diagonalizable and semi-Hamiltonian. By imposing a further constraint, the compatibility is reduced to a system of algebraic equations, whose solutions are described.

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1. Introduction

Dispersionless integrable equations arise in various contexts and have attracted people to investigate it from different point of view, such as topological field theory (WDVV equation) [3, 13–15, 30], matrix models [6, 7], conformal maps and interface dynamics [32, 40, 41], Einstein–Weyl space [16, 17]. The hydrodynamic reductions are the most developed method to find the exact solutions of the dispersionless integrable equations [5, 8, 12, 21, 22, 28–30]. From the hydrodynamic reductions, one can construct the Riemann invariants and the corresponding characteristic speeds satisfy the semi-Hamiltonian property or Tsarev's condition [38]. Hence, the generalized hodograph method can be used to find the exact solutions. Also, the solutions of dispersionless integrable equations can be found by the nonlinear Beltrami equation [26, 27], slightly different from the generalized hodograph method.

Let us recall the dispersionless non-standard Lax hierarchy [33, 39]. Suppose that λ is an algebra of the Laurent series of the form

$$\lambda = \left\{ A | A = \sum_{i=-\infty}^{N} a_i p^i \right\},$$

with coefficients a_i depending on an infinite set of variables $t_1 \equiv x, t_2, t_3, \ldots$. We can define a Lie bracket associated with λ as follows:

$$\{A, B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}, \qquad A, B \in \lambda$$

which can be regarded as the Poisson bracket defined in the two-dimensional phase space (x, p). The algebra λ can be decomposed into the Lie sub-algebras as

$$\lambda = \lambda_{\geqslant k} \oplus \lambda_{<k} \qquad (k = 0, 1, 2),$$

where

$$\lambda_{\geqslant k} = \left\{ A \in \Lambda | A = \sum_{i \geqslant k} a_i p^i \right\}$$

$$\lambda_{<k} = \left\{ A \in \lambda | A = \sum_{i<k} a_i p^i \right\}.$$

Based on this, the Lax formulation of the dispersionless integrable hierarchy can be formally defined as

$$\frac{\partial \lambda}{\partial t_n} = \left\{ \left(\lambda^{\frac{n}{N}}\right)_{\geqslant k}, \lambda \right\}.$$

- For $k = 0$, it is called the dispersionless KP hierarchy (dKP) [39].
- For $k = 1$, it is called the dispersionless modified KP (dmKP) hierarchy [11, 34].
- For $k = 2$, it is called the dispersionless Harry Dym (dDym) hierarchy [11, 12, 34]. It is the purpose of this paper.

We define the dispersionless Harry Dym (dDym) hydrodynamic systems as follows. The Lax operator of dDym has the form

$$\lambda(p) = A_{-1}p + A_0 + A_1 p^{-1} + A_2 p^{-2} + A_3 p^{-3} + \cdots.$$

Then the dDym hydrodynamic system is [11, 34]

$$\frac{\partial \lambda}{\partial t_n} = \{\lambda, \Omega_n(p)\}, \tag{1}$$

where

$$\Omega_n(p) = (\lambda(p)^n)_{\geqslant 2}.$$

Here $(\,)_{\geqslant 2}$ denotes the projection of the Laurent series onto a linear combination of $\lambda(p)^n$ with $n \geqslant 2$. From the zero curvature equation ($t_3 = t$ and $t_2 = y$)

$$\frac{\partial \Omega_2(p)}{\partial t} - \frac{\partial \Omega_3(p)}{\partial y} = \{\Omega_2(p), \Omega_3(p)\},$$

where

$$\Omega_2(p) = A_{-1}^2 p^2 \qquad \text{and} \qquad \Omega_3(p) = A_{-1}^3 p^3 + 3A_{-1}^2 A_0 p^2,$$

one can get the dispersionless Harry Dym equation [11]

$$\frac{\partial A_{-1}}{\partial t} = \frac{3}{4}\frac{1}{A_{-1}}\left[A_{-1}^2 \partial_x^{-1}\left(\frac{A_{-1y}}{A_{-1}^2}\right)\right]_y.$$

Now, considering the y-flow, we have

$$\lambda_y = \left\{\lambda, A_{-1}^2 p^2\right\} = 2pA_{-1}^2 \lambda_x - \left(A_{-1}^2\right)_x p^2 \lambda_p, \tag{2}$$

or

$$
\begin{aligned}
A_{-1y} &= 2A_{-1}^2 A_{0x} \\
A_{0y} &= 2A_{-1}^2 A_{1x} + \left(A_{-1}^2\right)_x A_1 \\
&\;\;\vdots \\
A_{ny} &= 2A_{-1}^2 A_{n+1,x} + (n+1)\left(A_{-1}^2\right)_x A_{n+1},
\end{aligned}
\tag{3}
$$

where $n = -1, 0, 1, 2, 3, \ldots$. Comparing it with the Benney moment chain [23, 36, 44], one calls (3) the dDym moment chain. It is the subject of this paper.

The paper is organized as follows. In the next section, one considers the reduction problems and obtains finitely many partial differential equations. In section 3, we prove the semi-Hamiltonian property when using Riemann invariants. In section 4, by imposing a further constraint, one gets some particular reductions. In the final section, we discuss some problems to be investigated.

2. The compatibility conditions

In this section, we consider the hydrodynamic reduction problems following [1, 21, 22]. For non-hydrodynamic reductions, one refers to [2].

We assume that the moments A_i are functions of only N independent variables u_i. If the A_i satisfy (3), then it is straightforward to show that the mapping

$$(u_{-1}, u_0, u_1, u_2, \ldots, u_{N-2}) \to (A_{-1}, A_0, A_1, A_2, \ldots, A_{N-2})$$

is non-degenerate. Hence without loss of generality, we set $u_{-1} = A_{-1}, u_0 = A_0, u_1 = A_1, \ldots, u_{N-2} = A_{N-2}$. The first N moments are the independent variables, while the higher moments are functions of them, i.e.,

$$A_k = A_k(A_{-1}, A_0, \ldots, A_{N-2}), \qquad k \geqslant N-1.$$

The equations of motion for $A_{-1}, A_0, \ldots, A_{N-2}$ become

$$\frac{\partial A_j}{\partial y} = 2A_{-1}^2 A_{j+1,x} + (j+1)\left(A_{-1}^2\right)_x A_{j+1}, \qquad j \leqslant N-3 \tag{4}$$

$$
\begin{aligned}
\frac{\partial A_{N-2}}{\partial y} &= 2A_{-1}^2 A_{N-1,x} + (N-1)\left(A_{-1}^2\right)_x A_{N-1} \\
&= 2A_{-1}^2 \frac{\partial A_{N-1}}{\partial A_j}\frac{\partial A_j}{\partial x} + (N-1)\left(A_{-1}^2\right)_x A_{N-1},
\end{aligned}
\tag{5}
$$

while each higher moment $(A_{N-1}, \ldots,)$ must satisfy the overdetermined system $(k \geqslant N-1)$ using (4), (5)

$$
\begin{aligned}
\frac{\partial A_k}{\partial y} &= \sum_{j=-1}^{N-3} \frac{\partial A_k}{\partial A_j}\left[2A_{-1}^2 A_{j+1,x} + (j+1)\left(A_{-1}^2\right)_x A_{j+1}\right] \\
&\quad + \frac{\partial A_k}{\partial A_{N-2}}\left[2A_{-1}^2 \sum_{j=-1}^{N-2} \frac{\partial A_{N-1}}{\partial A_j}\frac{\partial A_j}{\partial x} + (N-1)\left(A_{-1}^2\right)_x A_{N-1}\right] \\
&= 2A_{-1}^2\left[\sum_{j=-1}^{N-2} \frac{\partial A_{k+1}}{\partial A_j}\frac{\partial A_j}{\partial x}\right] + (k+1)\left(A_{-1}^2\right)_x A_{k+1}.
\end{aligned}
$$

Comparing the coefficients of $\frac{\partial A_j}{\partial x}$ $(j = -1, 0, 1, \ldots, N-2)$, one has

$$\frac{\partial A_{k+1}}{\partial A_j} = \frac{\partial A_k}{\partial A_{j-1}} + \frac{\partial A_k}{\partial A_{N-2}} \frac{\partial A_{N-1}}{\partial A_j}, \qquad 0 \leqslant j \leqslant N-2 \tag{6}$$

$$A_{-1}\frac{\partial A_{k+1}}{\partial A_{-1}} = \sum_{j=-1}^{N-2} \frac{\partial A_k}{\partial A_j}(j+1)A_{j+1} + A_{-1}\frac{\partial A_k}{\partial A_{N-2}} \frac{\partial A_{N-1}}{\partial A_{-1}} - (k+1)A_{k+1}. \tag{7}$$

Now, letting $k = N-1$ and defining $r = \log A_{-1}$, one has

$$\frac{\partial A_N}{\partial A_j} = \frac{\partial A_{N-1}}{\partial A_{j-1}} + \frac{\partial A_{N-1}}{\partial A_{N-2}} \frac{\partial A_{N-1}}{\partial A_j}, \qquad 0 \leqslant j \leqslant N-2 \tag{8}$$

$$\frac{\partial A_N}{\partial r} = \sum_{j=-1}^{N-2} \frac{\partial A_{N-1}}{\partial A_j}(j+1)A_{j+1} + \frac{\partial A_{N-1}}{\partial A_{N-2}} \frac{\partial A_{N-1}}{\partial r} - NA_N. \tag{9}$$

The compatibility of (8) and (9) gives a system Γ of $\frac{N(N-1)}{2}$ nonlinear second-order equation for the single known $A_{N-1}(A_{-1}, A_0, A_1, \ldots, A_{N-2})$. One can show that by induction if Γ is satisfied then the analogous compatibility for A_k $(k \geqslant N)$ is also derived. Let us investigate the case $N = 2$ in more detail. Then we have $(A_0 = s)$

$$\frac{\partial A_2}{\partial s} = A_{1r}\exp(-r) + A_{1s}^2$$

$$\frac{\partial A_2}{\partial r} = A_1 A_{1s} + A_{1s}A_{1r} - 2A_2.$$

Cross-differentiating, one gets the quasi-linear second differential equation

$$A_{1rr}\exp(-r) + A_{1s}A_{1rs} - (A_1 + A_{1rs})A_{1ss} + A_{1r}\exp(-r) + (A_{1s})^2 = 0. \tag{10}$$

Letting

$$A_1 = a, \qquad A_{1r} = b, \qquad A_{1s} = c,$$

one can express (10) as the degenerate non-homogeneous hydrodynamic system

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -c\exp r & (a+b)\exp r \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_s + \begin{pmatrix} b \\ -b - c^2\exp r \\ 0 \end{pmatrix}. \tag{11}$$

Define the characteristic speeds as

$$u = \frac{-c\exp r + \sqrt{c^2\exp(2r) + 4(a+b)\exp r}}{2}$$

$$v = \frac{-c\exp r + \sqrt{c^2\exp(2r) - 4(a+b)\exp r}}{2}.$$

A simple calculation yields

$$\begin{pmatrix} u \\ v \end{pmatrix}_r = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_s + \begin{pmatrix} \frac{uv}{v-u} \\ \frac{uv}{u-v} \end{pmatrix}, \tag{12}$$

there being no a-term! It is of non-homogeneous hydrodynamic systems of Tsarev–Gibbons type [21, 18] and it has one obvious hydrodynamic-type conserved density $(u+v)$. Moreover, according to the theory of Poisson commuting Hamiltonians [18], one can also find a conserved density of first derivatives:

$$(u-v)\left[\left(\frac{u_s}{u}\right)^2 - \left(\frac{v_s}{v}\right)^2\right].$$

For a given affinor $v_i^j(u)$ of a hydrodynamic system

$$\frac{\partial u_i}{\partial y} = v_i^j(u)\frac{\partial u_j}{\partial x}, \tag{13}$$

one can define the Nijenhuis tensor $N_{ij}^k(u)$ [35]

$$N_{ij}^k = v_i^s\frac{\partial v_j^k}{\partial u_s} - v_j^s\frac{\partial v_i^k}{\partial u_s} + v_s^k\frac{\partial v_i^s}{\partial u_j} - v_s^k\frac{\partial v_j^s}{\partial u_i}.$$

Then we can define the corresponding Haantjes tensor $H_{jk}^i(u)$ [24]

$$H_{jk}^i = \left(N_{qp}^i v_k^q - N_{kp}^q v_q^i\right)v_j^p - \left(N_{qj}^p v_k^q - N_{kj}^q v_q^p\right)v_p^i.$$

The system (13) is diagonalized if and only if the Haantjes tensor $H_{jk}^i(u)$ vanishes identically and all the eigenvalues of the affinor $v_i^j(u)$ are real and distinct. That is, there exist N functions λ_n (Riemann invariants), depending on the variables u_i, in which equation (13) is diagonalized

$$\frac{\partial \lambda_n}{\partial y} = V_n\frac{\partial \lambda_n}{\partial x}, \tag{14}$$

where V_n are the eigenvalues of the matrix $v_i^j(u)$, called characteristic speed. For the hydrodynamic systems (4) and (5) of the reduced dDym, the Haantjes tensor $H_{jk}^i(u)$ vanishes identically whenever the systems (8) and (9) are satisfied. Therefore, any consistent reduction of dDym is diagonalizable and then we can use Riemann invariants to discuss the problem further.

Finally, one remarks that as in the case of dKP hierarchy [28], a similar argument shows that for the reduced dDym hierarchy (1) we also have the Kodama–Gibbons formulation: the Riemann invariants are

$$\lambda_n = \lambda(q_n), \qquad \text{where} \quad \frac{\partial \lambda}{\partial p}(q_n) = 0, \quad n = 1, 2, \ldots, N \tag{15}$$

and then the hierarchy (1) can be expressed as

$$\frac{\partial \lambda_n}{\partial t_m} = \hat{\Omega}_m(\hat{V}_n)\frac{\partial \lambda_n}{\partial x},$$

where $\hat{V}_n = (q_1, q_2, \ldots, q_N)$ and $\hat{\Omega}_m(\hat{V}_n) = \frac{\mathrm{d}\Omega_m(p)}{\mathrm{d}p}\Big|_{p=\hat{V}_n}$.

3. Semi-Hamiltonian property

In this section, one will prove the semi-Hamiltonian property (Tsarev's condition) [38] of the reduced dDym hierarchy (1) using Riemann invariants. Suppose that each moment A_n can be expressed as the Riemann's invariants λ_i, which satisfy the equation ($t_2 = y$)

$$\frac{\partial \lambda_i}{\partial y} = V_i\frac{\partial \lambda_i}{\partial x}, \tag{16}$$

where

$$V_i = 2A_{-1}^2 q_i, \qquad i = 1, 2, 3, \ldots, N.$$

Then the moment equations (3) can be written as

$$V_i A_{n,\lambda_i} = 2A_{-1}^2 A_{n+1,\lambda_i} + (n+1)A_{n+1}\left(A_{-1}^2\right)_{\lambda_i}. \tag{17}$$

On the other hand, from (1), we also have, equivalent to (17),

$$\frac{\partial \lambda}{\partial y} = \frac{\partial \lambda}{\partial p}\left(A_{-1}^2\right)_x p^2 - 2\frac{\partial \lambda}{\partial x}A_{-1}^2 p$$

and then, using (16), one obtains, after reshuffling terms,

$$\frac{\partial \lambda}{\partial \lambda_i} = p^2 \frac{\partial \lambda}{\partial p} (A_{-1}^2)_{\lambda_i} \frac{1}{V_i + 2A_{-1}^2 p}. \tag{18}$$

Cross-differentiating, one has ($\phi = A_{-1}^2$)

$$\begin{aligned} &\phi_{\lambda_i\lambda_j}(V_i + 2\phi p)(V_j + 2\phi p)^2 - \phi_{\lambda_i}\phi_{\lambda_j} 2\phi p^2 (V_i + 2\phi p) - (V_j + 2\phi p)^2 \phi_{\lambda_i} \frac{\partial V_i}{\partial \lambda_j} \\ &\quad - 2p\phi_{\lambda_i}\phi_{\lambda_j}(V_j + 2\phi p)^2 = \phi_{\lambda_i\lambda_j}(V_j + 2\phi p)(V_i + 2\phi p)^2 \\ &\quad - \phi_{\lambda_i}\phi_{\lambda_j} 2\phi p^2 (V_j + 2\phi p) - (V_i + 2\phi p)^2 \phi_{\lambda_j} \frac{\partial V_j}{\partial \lambda_i} - 2p\phi_{\lambda_i}\phi_{\lambda_j}(V_i + 2\phi p)^2. \end{aligned}$$

Letting $p = \frac{-V_j}{2\phi}$, we obtain

$$\frac{\partial V_j}{\partial \lambda_i} = \frac{1}{2} (\ln \phi)_{\lambda_i} \left[\frac{V_i V_j}{V_i - V_j} + V_j \right], \qquad i \neq j. \tag{19}$$

On the other hand, comparing the coefficients of p-power and using (19), we get the only equation

$$\phi_{\lambda_i\lambda_j} = \frac{\phi_{\lambda_i}\phi_{\lambda_j}}{\phi} \left[\frac{V_i V_j}{(V_j - V_i)^2} + 1 \right], \qquad i \neq j. \tag{20}$$

The higher moments A_n, with $n \geqslant 0$, can be solved recursively using (17). These equations (19), (20) are compatible and their solutions are parametrized by $2N$ functions of a single variable.

A direct calculation, using MAPLE, confirms that the reduced equation (16) is semi-Hamiltonian, that is,

$$\begin{aligned} \frac{\partial}{\partial \lambda_k} \left(\frac{\frac{\partial V_j}{\partial \lambda_i}}{V_j - V_i} \right) &= \frac{\partial}{\partial \lambda_k} \left(\frac{1}{2} (\ln \phi)_{\lambda_i} \left[1 - \left(\frac{V_i}{V_j - V_i} \right)^2 \right] \right) \\ &= \frac{\partial}{\partial \lambda_i} \left(\frac{1}{2} (\ln \phi)_{\lambda_k} \left[1 - \left(\frac{V_k}{V_j - V_k} \right)^2 \right] \right) \\ &= \frac{\partial}{\partial \lambda_i} \left(\frac{\frac{\partial V_j}{\partial \lambda_k}}{V_j - V_k} \right), \end{aligned}$$

for i, j, k all distinct. Then the reduced equations (16) are thus integrable by the generalized hodograph transformation [38].

4. Algebraic equations and special reductions

To investigate the reduction problems, we introduce $A = 2 \ln A_{-1}$ to put (19) and (20) in a more compact form ($i \neq j$)

$$\begin{aligned} \frac{\partial V_j}{\partial \lambda_i} &= \frac{1}{2} A_{\lambda_i} \left[\frac{V_i V_j}{V_i - V_j} + V_j \right] \\ A_{\lambda_i\lambda_j} &= A_{\lambda_i} A_{\lambda_j} \frac{V_i V_j}{(V_j - V_i)^2}, \end{aligned}$$

or, noting that $V_i = 2A_{-1}^2 q_i$,

$$\frac{\partial q_j}{\partial \lambda_i} = \frac{1}{2} A_{\lambda_i} \left[\frac{q_i q_j}{q_i - q_j} - q_j \right] \tag{21}$$

$$A_{\lambda_i \lambda_j} = A_{\lambda_i} A_{\lambda_j} \frac{q_i q_j}{(q_j - q_i)^2}. \tag{22}$$

Then as in [21] for the dKP case, we impose two further restrictions on the reduced system (21) and (22). First, from the form (21), we require that the reduced system is translation-invariant in the sense that

$$\lambda_i \to \lambda_i + c \Rightarrow q_i \to q_i, \qquad A \to A$$

or, equivalently,

$$\delta q_i = 0 \tag{23}$$

$$\delta A = 0, \tag{24}$$

where $\delta = \sum_{i=1}^N \frac{\partial}{\partial \lambda_i}$. Secondly, we require the homogeneity of the functions A and q_j in the variables λ_i. A should be of weight 0 and the q_j of weight -1. Hence,

$$R q_i = -\frac{1}{\kappa} q_i \tag{25}$$

$$R \frac{\partial A}{\partial \lambda_i} = -\frac{\partial A}{\partial \lambda_i}, \tag{26}$$

where κ is a positive integer and $R = \sum_{i=1}^N \lambda_i \frac{\partial}{\partial \lambda_i}$. Plugging (22) into (24), (26) and eliminating the second derivative, we obtain

$$\begin{aligned} -\frac{\partial A}{\partial \lambda_i} = R \frac{\partial A}{\partial \lambda_i} &= \sum_{j=1, j\neq i}^N \lambda_j \frac{\partial^2 A}{\partial \lambda_i \lambda_j} + \lambda_i \frac{\partial^2 A}{\partial \lambda_i^2} \\ &= \sum_{j=1, j\neq i}^N \lambda_j \frac{\partial^2 A}{\partial \lambda_i \lambda_j} + \lambda_i \left(-\sum_{j=1, j\neq i}^N \frac{\partial^2 A}{\partial \lambda_i \lambda_j} \right) \\ &= \sum_{j=1, j\neq i}^N (\lambda_j - \lambda_i) A_{\lambda_j} A_{\lambda_i} \frac{q_i q_j}{(q_j - q_i)^2}. \end{aligned}$$

Hence, either $A_{\lambda_i} = 0$ or

$$\sum_{j=1, j\neq i}^N (\lambda_j - \lambda_i) A_{\lambda_j} \frac{q_i q_j}{(q_j - q_i)^2} = -1. \tag{27}$$

Similarly, plugging (21) into (23) and (25), we get

$$\begin{aligned} -\frac{1}{\kappa} q_i = R q_i &= \sum_{j=1, j\neq i}^N \lambda_j \frac{\partial q_i}{\partial \lambda_j} + \lambda_i \frac{\partial q_i}{\partial \lambda_i} \\ &= \sum_{j=1, j\neq i}^N \lambda_j \frac{\partial q_i}{\partial \lambda_j} - \lambda_i \sum_{j=1, j\neq i}^N \frac{\partial q_i}{\partial \lambda_j} \end{aligned}$$

$$= \sum_{j=1, j\neq i}^{N} (\lambda_j - \lambda_i) \frac{\partial q_i}{\partial \lambda_j}$$

$$= \frac{1}{2} \sum_{j=1, j\neq i}^{N} (\lambda_j - \lambda_i) \frac{\partial A}{\partial \lambda_j} \left[\frac{q_i q_j}{q_j - q_i} - q_i \right]. \tag{28}$$

The systems (27) and (28) form $2N$ algebraic equations for $2N$ unknowns, the q_i and $\frac{\partial A}{\partial \lambda_i}$. The solutions of the system are essentially unique. With two Riemann invariants and $\kappa = 1$, a simple calculation can yield by carefully choosing the integration constants

$$q_1 = -q_2 = \frac{4}{\lambda_1 - \lambda_2}, \qquad A_{-1} = \frac{(\lambda_1 - \lambda_2)^2}{16}. \tag{29}$$

Hence from (16), we have

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}_y = \begin{pmatrix} \frac{(\lambda_1-\lambda_2)^3}{32} & 0 \\ 0 & -\frac{(\lambda_1-\lambda_2)^3}{32} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}_x. \tag{30}$$

This equation (30) will correspond to the reduction of the Lax operator constructed by the Riemann–Hilbert method [11, 12]

$$\lambda(p) = A_{-1} p + A_0 + \frac{1}{p}. \tag{31}$$

If we consider more general Lax reductions of the form for positive integers m and κ [11, 13]

$$\lambda(p)^m = A_{-1}^m p^m + w_{m-1} p^{m-1} + w_{m-2} p^{m-2} + \cdots + w_{-\kappa+1} p^{-\kappa+1} + p^{-\kappa}, \qquad m + \kappa = N, \tag{32}$$

then from (15) it is not difficult to see that (32) satisfies the conditions (23)–(26). Hence, its corresponding characteristic speeds and A_{-1} will be one of the solutions of the $2N$ algebraic equations (27) and (28). It would be interesting to know whether any other solutions exist.

Next, one generalizes the operator R in (25) or (26) to the following forms:

$$\hat{R} = \sum_{k=1}^{N} \hat{g}_k(\vec{\lambda}) \partial_{\lambda_k} \hat{h}_k(\vec{\lambda}) \qquad \hat{R} V_i = -\frac{1}{\kappa} V_i \tag{33}$$

and

$$\tilde{R} = \sum_{k=1}^{N} \tilde{g}_k(\vec{\lambda}) \partial_{\lambda_k} \tilde{h}_k(\vec{\lambda}) \qquad \tilde{R} \frac{\partial A}{\partial \lambda_i} = -\frac{\partial A}{\partial \lambda_i}, \tag{34}$$

where $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ and $(\hat{g}_k, \hat{h}_k), (\tilde{g}_k, \tilde{h}_k)$ are arbitrary functions satisfying the conditions compatible with (23), (24)

$$\delta \hat{g}_k = \delta \hat{h}_k = \delta \tilde{g}_k = \delta \tilde{h}_k = 0. \tag{35}$$

Then a similar calculation can get the following $2N$ algebraic equations generalizing (27), (28)

$$\sum_{j=1, j\neq i}^{N} (\hat{g}_j \hat{h}_j - \hat{g}_i \hat{h}_i) A_{\lambda_j} \left[\frac{q_j q_i}{q_j - q_i} + q_i \right] = -\frac{1}{\kappa} V_i \left(1 + \kappa \sum_{j=1}^{N} \hat{g}_j \frac{\partial \hat{h}_j}{\partial \lambda_j} \right) \tag{36}$$

$$\sum_{j=1, j\neq i}^{N} (\tilde{g}_j \tilde{h}_j - \tilde{g}_i \tilde{h}_i) A_{\lambda_j} \frac{q_i q_j}{(q_j - q_i)^2} = -\left(1 + \sum_{j=1}^{N} \tilde{g}_j \frac{\partial \tilde{h}_j}{\partial \lambda_j} \right). \tag{37}$$

For $N = 2$ and $\kappa = 1$, simple calculations can show that a suitable choice of $(\hat{g}_1, \hat{h}_1)$, $(\hat{g}_2, \hat{h}_2)$, $(\tilde{g}_1, \tilde{h}_1)$, $(\tilde{g}_2, \tilde{h}_2)$, not unique, can also obtain the solution (29).

Finally, one notices that if we let

$$\hat{h}_j = \tilde{h}_j = 1$$

for $j = 1, 2, \ldots, N$, then we get the weaker condition than (35)

$$\delta\hat{g}_k = \delta\tilde{g}_k.$$

Hence, equations (36), (37) reduce to

$$\sum_{j=1, j\neq i}^{N} (\hat{g}_j - \hat{g}_i) A_{\lambda_j} \left[\frac{q_j q_i}{q_j - q_i} + q_i \right] = -\frac{1}{\kappa} V_i$$

$$\sum_{j=1, j\neq i}^{N} (\tilde{g}_j - \tilde{g}_i) A_{\lambda_j} \frac{q_i q_j}{(q_j - q_i)^2} = -1.$$

If $\hat{g}_j = \tilde{g}_j = \lambda_j$, then we can obtain (27) and (28).

5. Concluding remarks

We prove the semi-Hamiltonian property of reductions for the dDym and find some particular solutions invariant under translation and homogeneity. In spite of the results obtained, there are some interesting issues deserving investigation.

- The integrability and solution structure of equation (12) (or (10)) is unclear [5, 18–20]. It is not difficult to see that (12) can be extended to

$$\frac{\partial u_i}{\partial r} = r_i \frac{\partial u_i}{\partial s} + \frac{u_1 u_2 \cdots u_N}{\Pi_{k\neq i}(u_k - u_i)}, \qquad \text{where} \quad r_i = \left(\sum_{k=1}^{N} u_k \right) - u_i.$$

  One hopes to address these problems elsewhere.
- In [31], the algebraic reductions for dKP are found and in [25, 43], the waterbag reduction (non-algebraic) for dKP is also found. If we define

$$A_n = \int_{-\infty}^{\infty} q^n f(q, x, y)\, \mathrm{d}q, \qquad n = -1, 0, 1, 2, \ldots$$

  then we obtain

$$\lambda = p^2 \left(P \int_{-\infty}^{\infty} \frac{f/q}{p - q} \mathrm{d}q \right), \qquad f = f(q, x, y), \tag{38}$$

  where $P\int$ denotes the Cauchy principal value of the integral. Also, from (38), the 'distribution' function $f(x, y, q)$ must satisfy the Vlasov-like equation [23, 37, 44]

$$f_y = 2A_{-1}^2 q f_x - \left(A_{-1}^2\right)_x q^2 f_q = \left\{ f, A_{-1}^2 q^2 \right\}_{x,q}. \tag{39}$$

  Comparing (39) and (2), we can assume $f = F(\lambda)$ for any function F. Hence, the Lax operator λ will satisfy the nonlinear singular integral equation

$$\lambda = p^2 \left(P \int_{-\infty}^{\infty} \frac{F(\lambda)/q}{p - q} \mathrm{d}q \right).$$

  This equation will help us find the (non-)algebraic reductions [22, 34] and study the initial value problem for dDym as in the case of dKP [42, 43]. It needs further investigations.

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